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TOWARD TO VIBRATION ANALYSIS BY KNOT THEORY

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***Abstract*— Vibration analysis is one of the most important aspects in the design of structures and mechanical systems, among others, subject to dynamic loads. As well as for the analysis of failures caused by vibratory aspects. A good performance of an industrial system is often associated with the availability of mathematical models of the dynamic behaviour of the system. In some situations, the complexity of the processes makes it difficult to have models that help us to analyse these processes. This paper proposes the use of knot theory, which is a topological tool, for vibration analysis. This topological tool, in this case, associates a topological invariant when there is a drastic change in vibrations. The present work is based on the fact that it is well known that the equations representing harmonic motion generate Lissajous figures. In knot theory, there are several classifications of knots, one of these classifications is known as Lissajous knots. The use of this tool is shown in the supposition that we have a system represented by three equations of the form $f(t) = A\cos(Bt + C)$, where with the indicated parameters it generates a knot (being its nominal value). Making a change in the phase, which represents a fault, generates a different knot than the nominal knot. One of the advantages of this proposed method is that it is not necessary to have the model, and one of the disadvantages by nature of this method is that three signals are required to use this topological tool.**

***Keywords*— Vibration Analysis, Knot Theory, Topological Invariant, Lissajous knots, Dynamical Systems.**

I. Introduction

Vibrations are one of the most common and important aspects of life. Natural phenomena, our own body, as well as mechanisms, involve some kind of vibratory motion [1],[2]. Robert Hooke was the first to discover and unveil vibratory motion by a vibrating glass plate [3]. From here, there were many pioneers in vibration analysis, such as Daniel Bernoulli, who was the first to formulate a differential equation for the vibrational motion of a beam. Leonhard Euler made many advances on the elastic curves through his investigations of the shape of elastic beams under various loading conditions [4]. From the investigations of Bernoulli and Euler, what is known as the Bernoulli-Euler beam theory is derived, this theory is commonly used to solve engineering problems. As well as, Pochhammer and Chree [5], who were the first to investigate for the first time an exact formulation of the beam problem in accordance with general elasticity equations. Joseph Fourier worked on the decomposition of periodic functions, which is an infinite sum of combinations of sines and cosines, this decomposition is known as Fourier series [6], [7]. The Fourier series is used to pass a signal in the time domain to the frequency domain and vice versa. This methodology is the most commonly used for vibration analysis.

Vibration analysis can be defined as a process of monitoring vibration levels and studying the patterns of those vibrations. This analysis is very important to measure the vibrations and frequencies of the machinery and then use that information to analyse the health status of the machines and their components [8], [9]. Machines are made up of various components that work together to achieve a certain objective. The vibrations generated in the machines come from each of its components. Given the complexity of most machines and their vibration signals, it is necessary to convert them into simpler signals in order to analyse and interpret their patterns [10], [11]. This is achieved by transforming the signal in time to the frequency domain through the Fast Fourier Transform (FFT). For practical reasons, industries use instruments that measure and analyse vibrations, which provide frequency spectra and the magnitude of their parameters.

When talking about vibration analysis, one of the important parameters is phase. Phase is a relative time difference between two signals measured in units of angle and not time [8], [9]. Phase measurement is used to decipher machine faults such as misalignment and unbalance. It only works if the two signals being compared have the same frequency. Here are some examples of how phase can help to analyse vibrations: Phase can be used to identify machine frame distortion, detect cocked bearings and bent shafts or detect loose joints on structures and bending or twisting due to weakness or resonance, among others [8], [9].

Knot theory is the branch of topology that studies the behaviour of three-dimensional structures without intersections and their invariants [12]. In the late 1970s and early 1980s, knot theory was introduced to the analysis of dynamical systems. R.F. Williams provides the concepts of knot theory to analyse the topological complexity of Lorenz attractor trajectories [13]. From the point of view of dynamical systems, a knot can be defined as a simple closed curve generated by a three-dimensional trajectory. Knots can be generated by a third-order ordinary differential equation, by the set of three equations representing a system, or by the set of three time series of a system [14], [15]. Topologically, the generated knots can be represented as a global solution of the system in R^3 [14], [15]. This theory aims to differentiate one knotted system from another, this tool makes it possible to know if two trajectories in R^3 have the same topological structure.

It is well known that if an object vibrates harmonically in two directions, plotting these two signals generates what are known as Lissajous figures. Lissajous figures are used to establish the frequency ratio or relative phase between two harmonic signals [16], [17]. Relative phase is the most practical way to measure phase on a machine [8], [9]. In knot theory, knots are classified in several ways, one of these classifications is known as Lissajous knots [18], [19]. This knots are produced by the set of three equations of the form $\mathrm{f(t)} = \mathrm{A}\cos(\mathrm{Bt}+\mathrm{C})$. In this work, the use of knot theory for the analysis of harmonic vibrations in three directions or dimensions is proposed. In this particular case, the use of this theory is shown, assuming that we have a system represented by three equations of the form $\mathrm{f(t)} = \mathrm{A}\cos(\mathrm{Bt}+\mathrm{C})$, where with the indicated parameters it generates a knot (being its nominal value). Making a change in the phase, which represents a fault, generates a different knot than the nominal knot.

II. Preliminaries

This section introduces the concepts that form the basis of the proposal of this work. It begins with a brief review of the important concepts of knot theory, as well as the concept of harmonic motion.

A. *Brief of Knot Theory*

Knot theory is the branch of topology that studies, among other things, closed trajectories in R^3. A knot <K> is an embedding $f: S^1 \rightarrow R^3$ that has no intersections in its closed trajectory [12], [18]. By definition, the knots are defined in R^3 however, to analyse and perform operation on them, the knots are projected in R^2. This projection is called *knot projection* and is represented as:

$$K : R^3 \longrightarrow R^2,$$
$$K(x, y, z) = K(x, y, 0).$$

Topology is a branch of math that studies the properties that remain invariant under smooth and continuous deformation. A topological invariant is defined as a quantity γ that remains unchanged; if here exists a homeomorphism of diffeomorphism that X and Y are topologically equivalent if $\gamma(X) = \gamma(Y)$ [20].

Knots can be deformed smoothly through Reidemeister's moves, which are shown in Fig. 1. These moves are isomorphism that modified the local geometry of the knot but remain the topology.

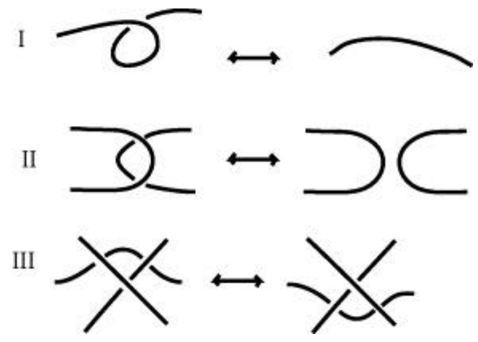

Fig. 1 Reidemeister's moves.

Reidemeister's moves indicate the steps to follow to know if two knots are equivalent but does not determine the required number of steps to perform it. A significance advance in this direction was the introduction of the *knot polynomial*. Two knots are topologically equivalent if they have the same knot polynomial [12], [18]. J.W. Alexander introduced the first knot polynomial, from here on new knot polynomials have been created, such as: Jone's polynomial, Kauffman's bracket, HOMFLY, among others. The knot polynomial is formed from the information of the knot crossings [21]. In this work, we will use the Alexander polynomial.

To calculate Alexander's polynomial [22], we start with an oriented diagram D of a knot K. Let v be the crossing points of the diagram: $c1, c2, \dots, cv$. By Euler's theorem, the arcs of the diagram D divide the plane into $v+2$ regions. Let rj, rk, rl y rm be the four regions surrounding the crossing point, a counter clockwise turn is made, according to Fig. 2 starting from the dotted regions which are rj y rk.

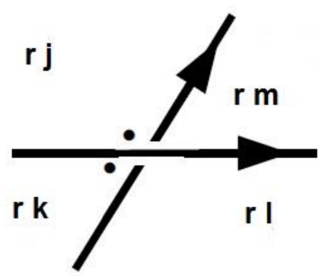

Fig. 2 Alexander's notation.

According to the above, the following linear equation can be defined:

$$ci(r) = trj - trk + rl - rm = 0, \qquad (1)$$

by taking an alternating sum of the symbols representing the four regions in their cyclic order and multiplying the dotted regions by t.

By defining an equation for each crossing of the diagram, we obtain a system of v equations in $v+2$ variables, which can be represented in a matrix $v(v+2)$, M, where each entry is $\pm t$, ± 1 o 0. In the matrix constructed, as described, each row of the matrix corresponds to cross points of the diagram and each column corresponds to regions. The next step in this process is to choose two neighbouring regions rp, rq and remove them from their respective columns vp , vq of the matrix. Eliminating the columns vp, vq we obtain a square matrix $v\ x\ v$, M_p, q. The matrix M_p, q is called the Alexander matrix of the knot K. Now let Δp, $q(t)$ be the determinant of this square matrix, which will be a polynomial in powers of t with integer coefficients.

Below is an example of Alexander's polynomial for knot 3_1 (trefoil knot). Considering the diagram of the trefoil knot in Fig. 3. Examining the crossing $c1$ we can see that the regions $r3$ and $r0$ are dotted and that the counterclockwise cycle is $r0$, $r3$, $r4$, $r1$.

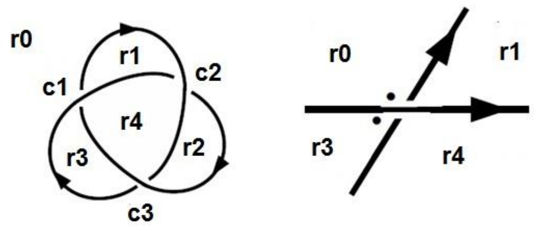

Fig. 3 Trefoil knot.

The equation obtained for the crossing $c1$ is:

$$c1(r) = tr0 - tr3 + r4 - r1 = 0, \quad (2)$$

Repeating the same process for the crossing points $c2$ and $c3$ gives us the following equations:

$$c2(r) = tr0 - tr1 + r4 - r2 = 0, \quad (3)$$

$$c3(r) = tr0 - tr2 + r4 - r3 = 0. \quad (4)$$

Now the above equations can be represented in the matrix:

$$A = \begin{pmatrix} t & -1 & 0 & -t & 1 \\ t & -t & -1 & 0 & 1 \\ t & 0 & -t & -1 & 1 \end{pmatrix}.$$

From the previous matrix we will eliminate two neighbouring regions which will be $r3$ and $r4$ which are the last two columns of the matrix and we will take the determinant of the matrix as $M_{\{3,4\}}$:

$$\triangle_{3,4}(t) = detM_{3,4} \begin{pmatrix} t & -1 & 0 \\ t & -t & -1 \\ t & 0 & -t \end{pmatrix} =$$

$$t \begin{pmatrix} -t & -1 \\ 0 & -t \end{pmatrix} + \begin{pmatrix} t & -1 \\ t & -1 \end{pmatrix},$$

$$= t^3 - t^2 + t,$$

$$= t(t^2 - t + 1).$$

Finally, we remove the factor of t from the polynomial and obtain the normalized polynomial:

$$\triangle k(t) = t^2 - t + 1.$$

This is the general procedure for calculating Alexander's polynomial, **KEBAP 3D** (KEBAP package, Software-Praktikums II group and Uni Hannover) is the software used for the Alexander polynomial calculation.

There are several classifications of knots, one of these classifications is known as Lissajous knots or Fourier knots. A Fourier knot is a knot that is represented by a parameterized curve in three-dimensional space, such that the function of the three coordinates of the curve are each finite parameters of the Fourier series. Such that, the knot can be considered as the result of independent vibrations in each of the coordinates and with each of these vibrations begins a linear combination of a finite number of pure frequencies [18], [19]. The Fourier series is an expression of the form [18], [19]:

$$f(t) = \sum_{i=0}^{\infty} A_i cos(B_i t + C_i), \quad (5)$$

where, for each term, $A_i \epsilon R$ is the amplitude, $B_i \epsilon Q$ is the frequency, and $C_i \epsilon\ R$ is the phase. Assume that we are given a knot in the form of a parametrised curve, such that each of the coordinate functions is smooth C^{∞}:

$$f : R \longrightarrow R^3, \qquad (6)$$
$$t \longrightarrow (x(t), y(t), z(t)).$$

Any smooth periodic function can be expressed as a Fourier series and can be parameterized as follows:

$$\begin{aligned} x(t) &= A_x cos(B_x t + C_x), \\ y(t) &= A_y cos(B_y t + C_y), \\ z(t) &= A_z cos(B_z t + C_z). \end{aligned} \qquad (7)$$

Knots that can be represented in this way are known as Lissajous knots. The amplitude (A) is only a scaling factor, if it changes it does not affect the topology of the knot. It is only the change in phases that matters.

The knot 3_1 is a peculiar Lissajous knot, which is represented by the following system of equations [18], [19]:

$$\begin{aligned} x &= cos(2t + 6), \\ y &= cos(3t + 0.15), \\ z &= cos(4t + 1) + cos(5t). \end{aligned} \qquad (8)$$

Fig. 4 shows the trefoil knot generated by the system of equations (8)

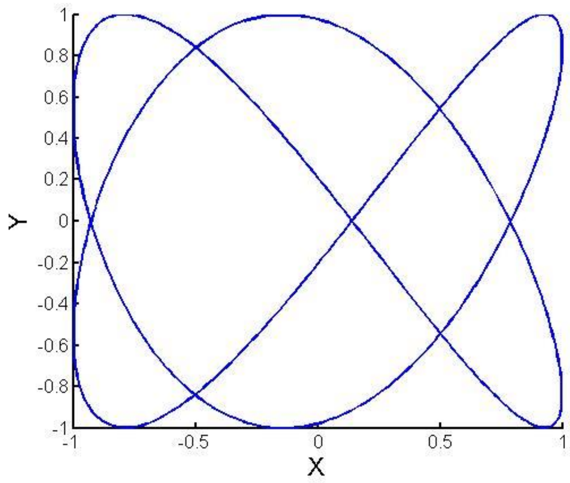

Fig. 4 Lissajous presentation of the trefoil knot

Fig. 5 shows the calculation of Alexander's polynomial of the knot of the Fig. 4 using the KEBAP 3D software.

B. *Vibration kinematics*

There are several vibratory motions, such as: harmonic motion, periodic motion, non-periodic motion, among others. In this paper, we will focus on harmonic motion. Harmonic motion is defined as a function of sines or cosines of the form [23], [24]:

$$x(t) = Acos(\omega t + \varphi), \qquad (9)$$

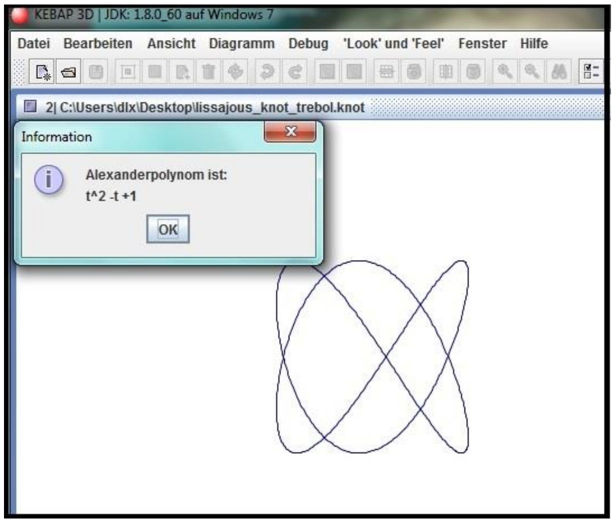

Fig. 5 Calculation of Alexander's polynomial of the knot of Fig. 4.

where A, ω, and φ are constants. The maximum displacement x_{max} with respect to the equilibrium position is called amplitude A. The argument of the cosine function, $\omega t + \varphi$, is called the *phase of motion* and the constant φ is called the *phase constant*. This constant corresponds to the phase when $t = 0$. The relationship between angular frequency ω , frequency f, and period of vibrations T is: $\omega = 2\pi f = 2\pi/T$. If we have two or more oscillating systems with equal amplitude and frequency, but different phase, we can choose $\varphi = 0$ for one of them. In this case, we assume that the object vibrates harmonically in three directions, as shown in the Fig. 6

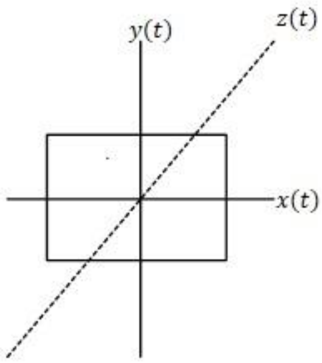

Fig. 6 Vibrations in three directions

Therefore, the Fig. 6 can be represented by the following system of equations:

$$
\begin{aligned}
x(t) &= Acos(\omega t + \varphi), \\
y(t) &= Acos(\omega t + \varphi), \\
z(t) &= Acos(\omega t + \varphi).
\end{aligned} \tag{10}
$$

As can be seen, the relationship between the system of equations (7) and (10) is obvious.

III. RESULTS

To show the proposed methodology, it is assumed that we have a system like the one in Fig. 6 which is represented by the system (7). With the following parameters, where $(B_x, B_y, B_z) = (2,3,7)$ and $(C_x, C_y, C_z) = (0.20,0.70,0)$ generate the knot 5_2, for this case, it is the nominal value of the system. Fig. 7 shows the knot 5_2, its Alexander polynomial is $2t^2 - 3t + 2$.

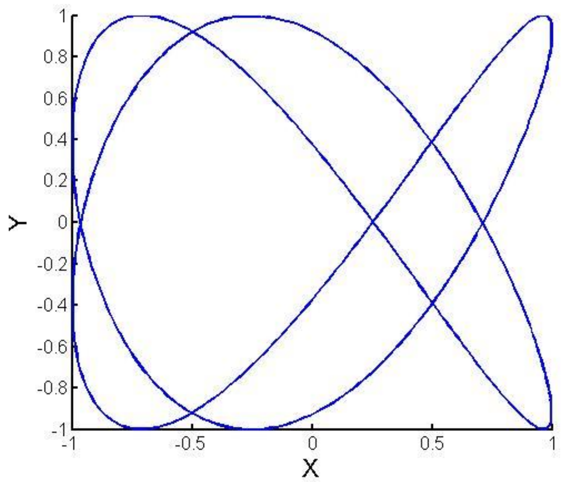

Fig. 7 knot 5_2 with (C_x, C_y, C_z) = (0.20,0.70,0).

Next, in Table I we present the results obtained by changing only the phases of the system (7) for certain values, which in this case simulate a system failure. According to the Table I, Fig. 8 shows the knot 0_1 which was generated with the phases $(C_x, C_y, C_z) = (0,30,45)$. Fig. 9 shows the knot 3_1 which was generated with the phases $(C_x, C_y, C_z) = (0,0,90)$. Fig.10 shows the knot 5_2 which was generated with the phases $(C_x, C_y, C_z) = (0,45,90)$. Finally, Fig.11 shows the knot 7_4 which was generated with the phases $(C_x, C_y, C_z) = (5,10,0)$.

TABLE I
RESULTS OBTAINED AT THE PHASE CHANGE OF SYSTEM (7)

(C_x, C_y, C_z)	Alexander polynomial
(0,30,45)	$1\ (0_1)$
(0,0,90)	$t^2 - t + 1\ (3_1)$
(0,45,90)	$2t^2 - 3t + 2\ \ (5_2)$
(5,10,0)	$4t^2 - 7t + 4\ \ (7_4)$

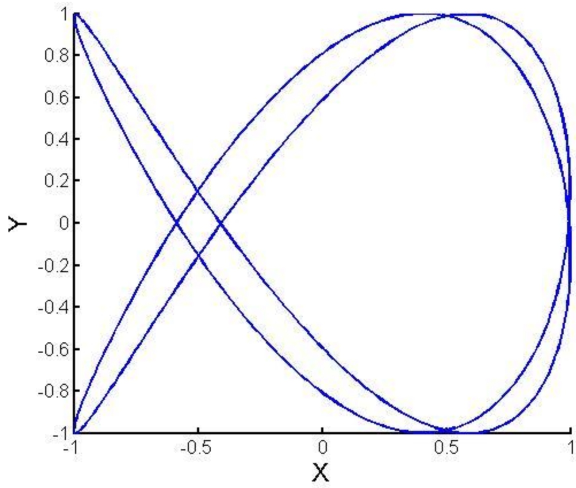

Fig. 8 knot 0_1 with (C_x, C_y, C_z) = (0,30,45).

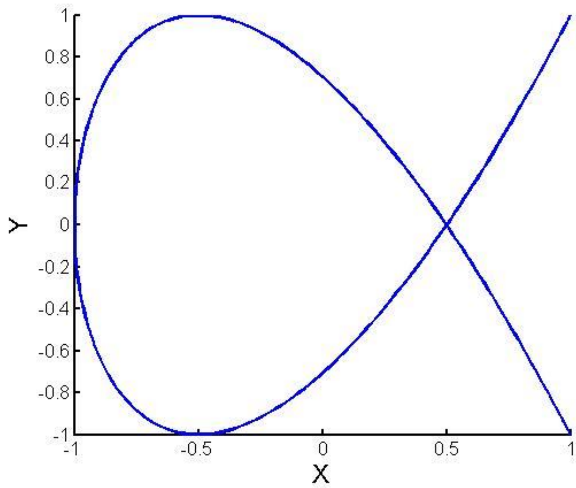

Fig. 9 knot 3_1 with (C_x, C_y, C_z) = (0,0,90).

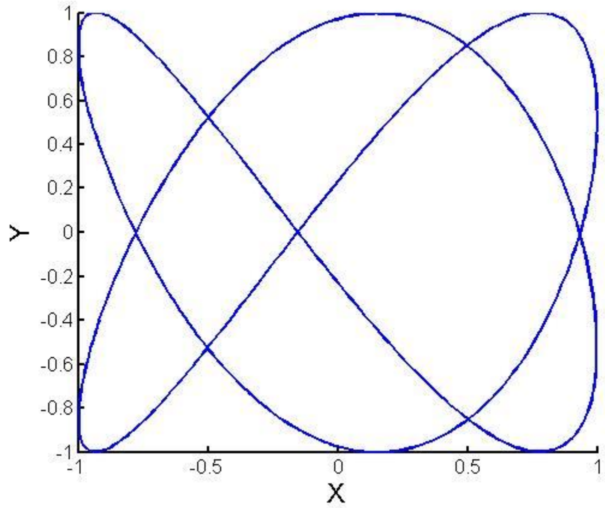

Fig. 10 knot 5_2 with $(C_x, C_y, C_z) = (0,45,90)$.

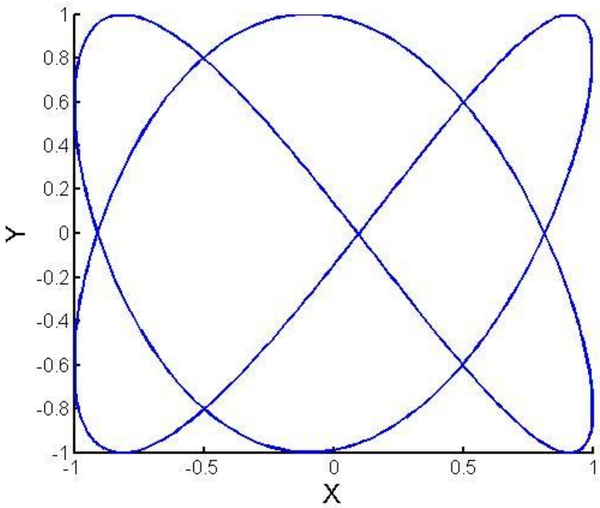

Fig. 11 knot 7_4 with $(C_x, C_y, C_z) = (5,10,0)$.

From the results, we have different knots with respect to the nominal value, indicating a failure for this case study. Except for Fig. 10, which represents the knot 5_2 being the same knot as the nominal value. As can be seen in Fig. 10 with respect to Fig. 7, the knot looks somewhat similar, but rotated 180 degrees, so this knot could be a *chiral knot* (a knot that is not equivalent to its mirror image); so it could be a false result, since the Alexander polynomial cannot distinguish a knot from its mirror image. As one of the future works, is the use of another knot polynomial, which distinguishes chiral knots.

IV. Conclusions

According to the results, knot theory can identify phase shifts in harmonic vibrations, so it can be an additional tool to the existing ones for vibration fault analysis and detection. In the same way, the knot polynomial gives us information, from the topological point of view, the type of knot and the number of crossings that generate those vibratory patterns.

As future work we intend to use sensors, such as an accelerometer, to acquire the vibrations of a system and really see how feasible the use of knot theory is; as well as what other methods would be used for its application.

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